A unifying pair of Cournot-Nash equilibrium existence results

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Pluralitas non est ponenda sine necessitate.

William of Ockham

For games with a measure space of players a tandem pair, consisting of a mixed and a pure Cournot-Nash equilibrium existence result, is presented. Their generality causes them to be completely mutually equivalent. This provides a unifying pair of Cournot-Nash existence results that goes considerably beyond the central result of [11, Theorem 2.1]. The versatility of this pair is demonstrated by the following new applications: (*i*) unification and generalization of the two equilibrium distribution existence results for anonymous games in [44], (*ii*) generalization of the equilibrium existence result for Bayesian differential information games in [38], (*iii*) inclusion of the Bayesian Nash equilibrium existence results in [41, 6] for games with private information in the sense of Harsanyi [33].

1 Introduction

In [11, Theorem 2.1] a central existence result was proposed for Cournot-Nash equilibria (CNE, for short) in (pseudo)games with a measure space of players. In the present paper this result will be considerably extended in the form of a tandem pair of CNE existence results, one mixed and one pure, that are mutually equivalent. Theorem 2.1.1, one half of this pair, is a mixed CNE existence result, just as Theorem 2.1 in [11], which it improves in several respects. It is paired with Theorem 2.2.1, an extension of a recent pure CNE existence result [15, Theorem 2.1] that is based on a new, so-called *feeble* topology. This topology owes its importance to the compactness conditions for player's action spaces that we impose throughout. In such a situation the feeble topology is a very flexible instrument: Examples 2.2.1 and 2.2.2 show that it *simultaneously* subsumes the two usual situations in the literature on games with a measure space of players, which work either with the weak topology $\sigma(L^1, L^\infty)$ or with its weak star counterpart $\sigma(L^\infty, L^1)$. As shown here, under those same compactness conditions for player's action spaces the feeble topology also subsumes the narrow topology that lies at the base of Theorem 2.1.1, the twin mixed CNE existence result mentioned before. This causes Theorems 2.1.1 and 2.2.1 to be *completely equivalent* (see Proposition 3.3.1), which testifies to the high level of generality of our results.

As a consequence of this generality, the present paper unifies CNE existence in the following three areas: (1) games with a measure space of players, as used by Schmeidler and others, (2) anonymous games *à la* Mas-Colell and (3) games with private information in the sense of Harsanyi and others.

This is not only evident from the applications already given in [9, 11, 13, 15] (these now also follow *a fortiori* from our central pair and will not be repeated here), but also from the following new applications in quite different directions: First, Theorem 2.1.1 is used to generalize and unify two separate CNE distribution existence results obtained by Rath in [44] (these in turn generalize [39]). This continues the approach to CNE equilibrium distributions given in [9, 11, 13], by which CNE distributions are seen as special mixed CNE's (i.e., they correspond to a special externality). Secondly, a generalization of the main existence result for Bayesian pure CNE's of Kim and Yannelis [38], who use a model with priors regarding interim beliefs, is given by means of Theorem 2.2.1.

A complete reduction of their model to the (pseudo)game form used in this paper is achieved via the formulation of a suitable "global" σ-algebra that keeps track of differential information. Thirdly, Theorem 2.2.1 is shown to generalize the existence results in [6, 21]. Those results, in turn, extended the well-known Bayesian CNE existence result of Milgrom-Weber [41] for games with private information *à la* Harsanyi [33] to the more natural situation where players' type spaces are non-topological.

In forthcoming work the methods of this paper will be devoted to a very general treatment of upper semicontinuity of the CNE correspondence [23].

2 Cournot-Nash equilibrium existence results

This section presents a tandem pair of mixed/pure Cournot-Nash equilibrium (CNE) existence results for continuum pseudogames, that is to say, pseudogames with an abstract measure space of players. In subsection 2.1 Theorem 2.1.1, the mixed CNE existence result, is formulated for a pseudogame Γ. Theorem 2.2.1, the pure CNE existence result, is given in subsection 2.2 for a pseudogame Γ'.

Common elements of the pseudogames Γ and Γ' are as follows. Both have a separable complete measure space $(T, \mathcal{T}, \mu)$ of *players* (or, if so desired, *player's types*); thus, the models are in the spirit of Aumann and Schmeidler [3, 45]. Recall that $(T, \mathcal{T}, \mu)$ is said to be separable if the (prequotient) space $\mathcal{L}^1(T, \mathcal{T}, \mu)$ is separable for the usual $\mathcal{L}^1$-seminorm. This separability assumption plays an important part in the proofs below (essentially, by allowing sequential arguments, which is sometimes very critical in measure theory). However, by exploiting a trick based on a result of Castaing and Valadier [25, p. 78], the separability assumption can be removed from all the existence results below – as opposed to their proofs – at the cost of only a slight strenghtening of the measurability conditions. Details about this trick can be found in [12] and in [15, Remark 4.2]. The present paper also extensively discusses it, but only in connection with (step 2 of) the proof of Theorem 4.4.1. The completeness assumption for $(T, \mathcal{T}, \mu)$ can be removed from the existence results as well. This goes by well-known reasoning involving measurable modifications, based on the fact that the central existence results allow for an exceptional null set (see [15, Remark 4.2] again). Both Γ and Γ' have for each player t a set S_t of *(individually) feasible actions.* All sets S_t, $t \in T$, are supposed to lie in an *action universe* S.

2.1 Mixed Cournot-Nash equilibrium existence result

This subsection centers around Theorem 2.1.1, a mixed CNE existence result for the pseudogame $\Gamma := (S_t, A_t, U_t)_{t \in T}$. The following assumptions must hold:

Assumption 2.1.1 *S is a completely regular Suslin space.*

Recall that a topological space is said to be *Suslin* if it is a topological Hausdorff space that is the surjective image of a Polish space under a continuous mapping; cf. [28, III], [46, II]. For instance, any Polish space S (i.e., a separable metric and complete space) or any Borel subset S of a Polish space meets the above assumption, and it continues to do so when equipped with a completely regular topology that is coarser than the original one. E.g., a separable Banach space meets Assumption 2.1.1, both for the norm-topology, for which it is a Polish space, and for the usual weak topology. Other examples include spaces that are countable unions of Polish spaces, such as the dual of a separable Banach space, when equipped with the weak star topology.

Assumption 2.1.2 (i) *For every $t \in T$ the set $S_t \subset S$ is nonempty and compact.*
(ii) $\text{gph}\, \Sigma \in \mathcal{T} \times \mathcal{B}(S)$.

Here $\Sigma : T \to 2^S$ is defined by $\Sigma(t) := S_t$ and its *graph* is given by $\text{gph}\, \Sigma := \{(t, s) \in T \times S : s \in S_t\}$. As usual, the symbol $\mathcal{B}(S)$ refers to the *Borel* σ-algebra on S and $\mathcal{T} \times \mathcal{B}(S)$ denotes the product σ-algebra. The trace of the latter σ-algebra on $\text{gph}\, \Sigma$ is denoted by $(\mathcal{T} \times \mathcal{B}(S)) \cap \text{gph}\, \Sigma$.

By Assumption 2.1.1, S has metric ρ that is not finer than its original topology (apply [25, III.32] or [28, III.66] – see [15, section 3] for an explicit description). Hence, Assumption 2.1.2(i) ensures that on the compact sets S_t, $t \in T$, these two topologies coincide. In other words: one might suppose just as well that the original topology on S is metrizable to begin with. Let us do this from now on.

The *mixed action universe* of Γ is the set $M_1^+(S)$, consisting of all probability measures on $(S, \mathcal{B}(S))$. This set is equipped with the classical narrow topology; cf. [24], [28, III]. The canonical *mixed action profiles* of Γ are the functions $\delta : T \to M_1^+(S)$, measurable with respect to $\mathcal{T}$ and $\mathcal{B}(M_1^+(S))$. Such δ's can be seen as descriptions/prescriptions of how all the players could or should act (in a mixed way) in the game. The set of all such mixed action profiles is denoted by $\mathcal{R}$. A mixed action profile $\delta \in \mathcal{R}$ is said to be *feasible* if $\delta(t)(S_t) = 1$ for a.e. (meaning μ-almost every) t in T; note carefully that there is an exceptional null set involved in this definition. The set of all such feasible profiles is denoted by $\mathcal{R}_\Sigma$.

Observe that Assumptions 2.1.1, 2.1.2 entail that $\mathcal{R}_\Sigma$ is nonempty. Indeed, the von Neumann-Aumann measurable selection theorem [25, III.22] can be applied here. This gives the existence of a function $f : T \to S$, measurable with respect to $\mathcal{T}$ and $\mathcal{B}(S)$, such that $f(t) \in S_t$ for a.e. $t \in T$; hence setting $\delta(t) := \epsilon_f(t) :=$ Dirac point measure at $f(t)$ defines a feasible mixed action profile.

For a proper understanding of the following topologization of $\mathcal{R}$, attention is called to the fact that, mathematically speaking, the mixed action profiles in $\mathcal{R}$ are precisely transition probabilities with respect to $(T, \mathcal{T})$ and $(S, \mathcal{B}(S))$ in the sense of [42, III] (see also [1, 2.6]); here the earlier observation about the metric ρ on S is instrumental. In one direction this is by [42, Proposition III.2.1] or [1, 2.6.4]; the other direction goes by Baire approximation [1, A6.6] and a Dynkin class argument [1, 4.1.2]. In connection with the following topology the elements of $\mathcal{R}$ are also often referred to as *Young measures*. Recall from [5, 6, 7] that the *narrow* topology on $\mathcal{R}$ (and on its subset $\mathcal{R}_\Sigma$) is defined as the coarsest topology on $\mathcal{R}$ for which all functionals

$$I_g : \delta \mapsto \int_T [\int_S g(t,s)\delta(t)(ds)]\mu(dt), g \in \mathcal{G}_C(T; S),$$

are continuous. Note that those integrals are well-defined by [42, III]. Here $\mathcal{G}_C(T; S)$ stands for the collection of all *Carathéodory integrands* on $T \times S$. Recall that this is the set of all $\mathcal{T} \times \mathcal{B}(S)$-measurable functions $g : T \times S \to \mathbb{R}$ for which $g(t, \cdot)$ is continuous on S for every $t \in T$ and for which there is an integrable function $\phi_g \in \mathcal{L}^1_{\mathbb{R}}(T, \mathcal{T}, \mu)$ with $\sup_{s \in S} |g(t,s)| \leq \phi_g(t)$ for all $t \in T$. Equivalently (apply [6, Theorem 2.2]), the narrow topology on $\mathcal{R}$ is the coarsest topology for which all functionals

$$I_g : \delta \mapsto \int_T [\int_S g(t,s)\delta(t)(ds)]\mu(dt), g \in \mathcal{G}^{bb}(T; S),$$

are lower semicontinuous. Here $\mathcal{G}^{bb}(T; S)$ is the collection of all *normal integrands* on $T \times S$ that are *integrably bounded below*; these are the $T \times \mathcal{B}(S)$-measurable functions $g : T \times S \to \mathbb{R}$ such that $g(t, \cdot)$ is lower semicontinuous on S for every $t \in T$ and for which there is an integrable function $\phi_g \in \mathcal{L}^1_{\mathbb{R}}(T, \mathcal{T}, \mu)$ with $\inf_{s \in S} g(t,s) \geq \phi_g(t)$ for all $t \in T$. Evidently, the narrow topology on $M_1^+(S)$, to which reference was already made, can be seen as a special case of the above narrow topology on $\mathcal{R}$ (e.g., consider what happens to the *constant* mixed action profiles or what happens when T is a singleton). To distinguish it from the latter, it will from now on consistently be called the *classical* narrow topology, In connection with subsection 3.3, the following addition fact is useful: The restriction of the narrow topology to $\mathcal{R}_\Sigma \subset \mathcal{R}$ is precisely the coarsest topology for which all functionals

$$I_g : \delta \mapsto \int_T [\int_S g(t,s)\delta(t)(ds)]\mu(dt), g \in \mathcal{G}_{C,\Sigma}(T),$$

are continuous on $\mathcal{R}_\Sigma$. Here $\mathcal{G}_{C,\Sigma}(T)$ is the set of all $(\mathcal{T} \times \mathcal{B}(S)) \cap \text{gph } \Sigma$-measurable functions $g : \text{gph } \Sigma \to \mathbb{R}$ for which $g(t, \cdot)$ is continuous on S_t for every $t \in T$ and for which there exists an integrable function $\phi_g \in \mathcal{L}^1_{\mathbb{R}}(T, \mathcal{T}, \mu)$ with $\sup_{s \in S_t} |g(t,s)| \leq \phi_g(t)$ for all $t \in T$. This is a direct consequence of the above equivalence: one has $g_1, g_2 \in \mathcal{G}^{bb}(T; S)$ by setting $g_i := (-1)^i g$ on gph Σ and $g_i := +\infty$ on $(T \times S) \backslash \text{gph } \Sigma$, with $I_{g_1}(\delta) = -I_{g_2}(\delta)$ for all $\delta \in \mathcal{R}_\Sigma$.

As a social feature of Γ, each player must choose his/her actions in accordance with the other players as follows: given the profile $\delta \in \mathcal{R}_\Sigma$, player t's *socially feasible* actions constitute a given subset $A_t(\delta) \subset S_t$. In a truly noncooperative situation one can of course eliminate such social interaction by choosing

$$A_t(\delta) := S_t \text{ for all } t \in T \text{ and } \delta \in \mathcal{R}_\Sigma. \tag{2.1}$$

Assumption 2.1.3 *(i) For every $(t, \delta) \in T \times \mathcal{R}_\Sigma$ the set $A_t(\delta) \subset S_t$ is nonempty and closed.*
(ii) For every $t \in T$ the multifunction $A_t : \mathcal{R}_\Sigma \to 2^{S_t}$ is (narrowly) upper semicontinuous.
(iii) For every $\delta \in \mathcal{R}_\Sigma$ the graph of the multifunction $t \mapsto A_t(\delta)$ belongs to $\mathcal{T} \times \mathcal{B}(S)$.

To measure the consequences of player t's actions in the face of his/her opponents, one introduces the *payoff function* $U_t : S_t \times \mathcal{R}_\Sigma \to [-\infty, +\infty]$. Given the mixed action profile $\delta \in \mathcal{R}_\Sigma$, player t receives $U_t(s, \delta)$ for taking action $s \in S_t$ (see also the comments following Theorem 2.1.1).

Assumption 2.1.4 *(i) For every $t \in T$ the function $U_t : S_t \times \mathcal{R}_\Sigma \to [-\infty, +\infty]$ is upper semicontinuous.*
(ii) For every $\delta \in \mathcal{R}_\Sigma$ the function $(t, s) \mapsto U_t(s, \delta)$ is $(\mathcal{T} \times \mathcal{B}(S)) \cap \text{gph } \Sigma$-measurable.

The final assumption requires a certain interplay between social feasibility and payoff:

Assumption 2.1.5 *For every $t \in T$ the function $\delta \mapsto \sup_{s \in A_t(\delta)} U_t(s, \delta)$ is (narrowly) lower semicontinuous.*

Remark 2.1.1 *(i) In the strictly noncooperative situation of (2.1) Assumption 2.1.3 holds automatically; also, in that situation Assumption 2.1.5 certainly holds if $U_t(s, \cdot)$ is narrowly lower semicontinuous on $\mathcal{R}_\Sigma$ for every $(t, s) \in \text{gph } \Sigma$ (of course, together with Assumption 2.1.4(i) this implies that $U_t(s, \cdot)$ is then narrowly continuous).*
(ii) In the terminology of the highly tautological [47, Proposition 1], Assumption 2.1.5 states that $U_t(s, \delta)$ is feasible path transfer lower semicontinuous in s with respect to A_t *for every $t \in T$.*
(iii) The measurability Assumptions 2.1.3(iii) and 2.1.4(ii) serve exclusively to make the graph of the multifunction $t \mapsto \text{argmax}_{s \in A_t(\delta)} U_t(s, \delta)$ $\mathcal{T} \times \mathcal{B}(S)$-measurable.

The main result of this subsection, a result about existence of a mixed Cournot-Nash equilibrium profile in Γ, can now be stated. Observe below that under such an equilibrium profile μ-almost every player t randomizes over actions that maximize his/her own payoff in a socially feasible way. The proof of this result will be given in subsection 3.1.

Theorem 2.1.1 (mixed equilibrium existence result) *Under the Assumptions 2.1.1 to 2.1.5 there exists a* mixed Cournot-Nash equilibrium *for the above pseudogame Γ. That is, there exists a mixed action profile $\delta_* \in \mathcal{R}_\Sigma$ such that*

$$\delta_*(t)(\text{argmax}_{s \in A_t(\delta_*)} U_t(s, \delta_*)) = 1 \textit{ for } \mu\textit{-a.e. } t \textit{ in } T.$$

This result improves Theorem 2.1, the main result of [11], in the following respects: (1) Assumption 2.1.5 improves upon the continuity requirement in [11, Assumption 2.6]; cf. Remark 2.1.1(*i*). (2) In [11] only the purely noncooperative situation with (2.1) is considered. (3) Theorem 2.1.1 deals directly with $U_t(s, \delta)$. In contrast, in [11] a $U_t(s, \delta)$ of the form $U_t(s, e_t(\delta))$ is used, with the technical complication that all mappings e_t, $t \in T$, on $\mathcal{R}_\Sigma$ should map into a common space that is itself Suslin and metric.

2.2 Pure Cournot-Nash equilibrium existence result

In this subsection a pure counterpart to the above existence result Theorem 2.1.1 is presented. This result (partially) allows for purification by nonatomicity. The counterpart to Γ is now a pseudogame $\Gamma' := (S_t, A'_t, U'_t)_{t \in T}$ in pure actions. Let us suppose that T is partitioned into two different groups of players, i.e., $T = \bar{T} \cup \hat{T}$ and $\bar{T} \cap \hat{T} = \emptyset$.

Assumption 2.2.1 *(i) $\bar{T}$, $\hat{T} \in \mathcal{T}$.*
(ii) $\hat{T}$ is contained in the nonatomic part of the measure space $(T, \mathcal{T}, \mu)$.

Purification by nonatomicity is to take place on the part $\hat{T}$.

Assumption 2.2.2 *S is a Suslin locally convex topological vector space.*

Observe that this assumption entails that S is completely regular as well, which makes it a specialization of Assumption 2.1.1. As before, we define $\Sigma : T \to 2^S$ by $\Sigma(t) := S_t$. and denote its graph by gph Σ.

Assumption 2.2.3 *(i) For every $t \in \bar{T}$ the set $S_t \subset S$ is nonempty, convex and compact.*
(ii) For every $t \in \hat{T}$ the set $S_t \subset S$ is nonempty and compact.
(iii) gph $\Sigma \in \mathcal{T} \times \mathcal{B}(S)$.

A *pure action profile* of Γ' is a function $f : T \to S$ that is measurable with respect to $\mathcal{T}$ and $\mathcal{B}(S)$ or, which is equivalent, for which all scalar functions $t \mapsto< f(t), s^* >$, $s^* \in S^*$, are $\mathcal{T}$-measurable. Here S^* stands for the topological dual of S. Such equivalence of ordinary and scalar measurability is due to the Suslin nature of S, which causes $\mathcal{B}(S)$ to coincide with the Borel σ-algebra for the weak topology $\sigma(S, S^*)$ (apply [46, Corollary 2, p. 101]). Let $\mathcal{S}$ denote the set of all such action profiles. A pure action profile $f \in \mathcal{S}$ is *feasible* if $f(t) \in S_t$ for μ-a.e. t in T. The set of all feasible action profiles is denoted by $\mathcal{S}_\Sigma$. Also, let $\bar{\mathcal{S}}_\Sigma$ be the set of all restrictions to $\bar{T}$ of functions in $\mathcal{S}_\Sigma$; it is only this set that needs to be topologized. Recall from [15] that the *feeble topology* on $\mathcal{S}_\Sigma$ is defined as the coarsest topology for which all functionals

$$J_g : f \mapsto \int_{\bar{T}} g(t, f(t))\mu(dt), \bar{g} \in \bar{\mathcal{G}}_{LC,\Sigma},$$

are continuous. Here $\bar{\mathcal{G}}_{LC,\Sigma}$ is the collection of all $(\mathcal{T} \cap \bar{T}) \times \mathcal{B}(S)$-measurable functions $g : \bar{T} \times S \to \mathbb{R}$ for which $g(t, \cdot)$ is linear and continuous on S for every $t \in \bar{T}$ and for which there there is an integrable function $\phi_g \in \mathcal{L}^1_{\mathbb{R}}(\bar{T}, \mathcal{T} \cap \bar{T}, \mu)$ with $\sup_{s \in S_t} |g(t, s)| \leq \phi_g(t)$ for all $t \in \bar{T}$. Note that this causes the above functional to be well-defined. In the special case $\bar{T} = T$ we shall write $\mathcal{G}_{LC,\Sigma}$ instead of $\bar{\mathcal{G}}_{LC,\Sigma}$. The following two examples show that, quite remarkably, the feeble topology can simultaneously subsume the two customary topologies that have been used in the literature on games with a measure space of players.

Example 2.2.1 Let S be a separable Banach space, equipped with either the norm topology or the weak topology $\sigma(S, S^*)$. In addition to what is required in Assumption 2.1.2, let $\Sigma : T \to 2^S$ be *integrably bounded*; that is to say, there exists $\phi_\Sigma \in \mathcal{L}^1_{\mathbb{R}}(T, \mathcal{T}, \mu)$ such that $\sup_{s \in S_t} \|s\| \leq \phi_\Sigma(t)$ for every $t \in T$. Here $\|\cdot\|$ stands for the norm on S. In this situation S is a Suslin locally convex topological vector space, and $\mathcal{S}_\Sigma$ is precisely the prequotient space $\mathcal{L}^1_\Sigma$ consisting of all Bochner-integrable μ-a.e.-selectors of the multifunction Σ. Also, on $\mathcal{S}_\Sigma = \mathcal{L}^1_\Sigma$ the feeble topology coincides in this situation with the usual (prequotient) weak $\mathcal{L}^1$-topology $\sigma(\mathcal{L}^1_S(T, \mathcal{T}, \mu), \mathcal{L}^\infty_{S^*}[S](T, \mathcal{T}, \mu)$. Recall here from [35, IV] that $\mathcal{L}^\infty_{S^*}[S](T, \mathcal{T}, \mu)$ is the space of all bounded and scalarly measurable functions from T into S^*, which can be identified with the dual of $\mathcal{L}^1_S(T, \mathcal{T}, \mu)$ for the usual $\mathcal{L}^1$-seminorm. The coincidence of these two topologies can be seen as follows. First, observe that on $\mathcal{S}_\Sigma$ the feeble topology is at least as fine as the weak $\mathcal{L}^1$-topology, simply because to every $b \in \mathcal{L}^\infty_{S^*}[S]$ there corresponds a canonical $g_b \in \mathcal{G}_{LC,\Sigma}$, given by $g_b(t, s) :=< s, b(t) >$ (observe that $\sup_{s \in S_t} |g_b(t, s)| \leq \phi_\Sigma(t) \text{ess sup}_T \|b(\cdot)\|^*$). Also, by [15, Proposition 3.2], which is a corollary of Theorems 3.1.1 and 3.2.2 used below, $\mathcal{S}_\Sigma$ is feebly compact. Unlike the feeble topology itself, the quotient of the feeble topology for the usual equivalence relation "equality μ-almost everywhere" is Hausdorff (denote this equivalence relation on the set of all measurable functions from T into S by π). So on the quotient-feebly compact set $\pi(\mathcal{S}_\Sigma)$ the quotient-feeble topology coincides with the usual quotient topology $\sigma(L^1_S, L^\infty_{S^*}[S])$. Since the defining functionals J_g, $g \in \mathcal{G}_{LC,\Sigma}$, for the feeble topology are constant on every π-equivalence class, it follows that the coincidence of these topologies can be carried back to the original prequotient setting.

In view of the above example, the referenced compactness result of [15, Proposition 3.2] can be considered as an extension of Diestel's theorem [48, Theorem 3.1]. Another situation considered on some occasions (e.g., cf. [36, p. 101]) is the following:

Example 2.2.2 Let S be the dual of a separable Banach space R and let S be equipped with the weak star topology $\sigma(S, R)$. Then S is the countable union of metrizable compacts (by the Alaoglu-Bourbaki theorem), whence a Suslin space. Following [36, p. 101], consider the situation where Assumption 2.2.3 holds and where all sets S_t, $t \in T$, are contained in a single dual norm-bounded set K. Then $\mathcal{S}_\Sigma$ is obviously the prequotient space $\mathcal{L}^\infty_\Sigma[R](T, \mathcal{T}, \mu)$ that consists of all bounded and R-scalarly measurable μ-a.e.-selectors of Σ. In this situation on $\mathcal{S}_\Sigma$ the feeble topology coincides with the weak star topology $\sigma(\mathcal{L}^\infty_S[R], \mathcal{L}^1_R)$. Notice that on $\mathcal{S}_\Sigma$ the feeble topology is at least as fine as the weak star topology, simply because to every $\ell \in \mathcal{L}^1_R$ there corresponds a canonical $g_\ell \in \mathcal{G}_{LC,\Sigma}$, given by $g_\ell(t, s) :=< \ell(t), s >$ (observe that $\sup_{s \in S_t} |g_\ell(t, s)| \leq r_K \|\ell(t)\|_R$, where r_K denotes the diameter of the set K). Again, the compactness result [15, Proposition 3.2] and a quotient argument can be used to show that these two topologies on $\mathcal{S}_\Sigma$ actually coincide.

Let us now define as the *externality* of each player $t \in T$ the mapping $d := (\bar{d}, \hat{d}) : \mathcal{S}_\Sigma \to \bar{\mathcal{S}}_\Sigma \times \mathbb{R}^m$, which is defined by

$$\bar{d}(f) := f \mid_{\bar{T}},\ \hat{d}(f) := (\int_{\hat{T}} g_i(t, f(t))\mu(dt))_{i=1}^m.$$

Here $f \mid_{\bar{T}} \in \bar{\mathcal{S}}_\Sigma$ stands for the restriction to $\bar{T}$ of $f \in \mathcal{S}_\Sigma$. Also, $g_1, \ldots, g_m : \mathrm{gph}\ \Sigma \cap (\hat{T} \times S) \to \mathbb{R}$ are given functions that satisfy the following condition.

Assumption 2.2.4 $g_1, \ldots g_m \in \mathcal{G}_{C,\Sigma}(\hat{T})$.

Thus, the externality d is such that on $\hat{T}$ the restriction $f \mid_{\hat{T}}$ of $f \in \mathcal{S}_\Sigma$, which completely describes the action $f(t)$ by each player t in $\hat{T}$, is replaced by the aggregate $\hat{d}(f)$ over all of $\hat{T}$. Observe that in the special situation with $\hat{T} := \emptyset$ and $\bar{T} := T$ we have $d(f) = f$ and $\bar{\mathcal{S}}_\Sigma = \mathcal{S}_\Sigma$. Each player $t \in T$ must choose his/her actions in accordance with the other players as follows: given the pure action profile $f \in \mathcal{S}_\Sigma$, player t's *socially feasible* actions constitute a given subset $A'_t(d(f)) \subset S_t$. Observe that the externality intervenes here. Of course, for a truly noncooperative situation one can always choose $A'_t \equiv S_t$, quite similar to (2.1). Further, every player $t \in T$ has a *payoff function* $U'_t : S_t \times \bar{\mathcal{S}}_\Sigma \times \mathbb{R}^m \to [-\infty, +\infty]$.

Assumption 2.2.5 *(i) For every $(t, \bar{f}, y) \in T \times \bar{\mathcal{S}}_\Sigma \times \mathbb{R}^m$ the set $A'_t(\bar{f}, y) \subset S_t$ is nonempty and closed.*
(ii) For every $t \in T$ the multifunction $A'_t : \bar{\mathcal{S}}_\Sigma \times \mathbb{R}^m \to 2^{S_t}$ is upper semicontinuous.
(iii) For every $(\bar{f}, y) \in \bar{\mathcal{S}}_\Sigma \times \mathbb{R}^m$ the graph of the multifunction $t \mapsto A'_t(\bar{f}, y)$ belongs to $\mathcal{T} \times \mathcal{B}(S)$.

Assumption 2.2.6 *(i) For every $t \in T$ the function $U'_t : S_t \times \bar{\mathcal{S}}_\Sigma \times \mathbb{R}^m \to [-\infty, +\infty]$ is upper semicontinuous.*
(ii) For every $(\bar{f}, y) \in \bar{\mathcal{S}}_\Sigma \times \mathbb{R}^m$ the function $(t, s) \mapsto U'_t(s, \bar{f}, y)$ is $(\mathcal{T} \times \mathcal{B}(S)) \cap \mathrm{gph}\ \Sigma$-measurable.

Again, the final assumption requires certain relationships between A' and U' to hold; this time, a convexity condition is added to what was required in the corresponding Assumption 2.1.5, but only for players in $\bar{T}$:

Assumption 2.2.7 *(i) For every $t \in T$ the function $(\bar{f}, y) \mapsto \sup_{s \in A'_t(\bar{f}, y)} U'_t(s, \bar{f}, y)$ is lower semi-continuous on $\bar{\mathcal{S}}_\Sigma \times \mathbb{R}^m$.*
(ii) For every $t \in \bar{T}$ and $(\bar{f}, y) \in \bar{\mathcal{S}}_\Sigma \times \mathbb{R}^m$ the set $\mathrm{argmax}_{s \in A'_t(\bar{f}, y)} U'_t(s, \bar{f}, y)$ is convex.

Of course, a counterpart to Remark 2.1.1 applies here:

Remark 2.2.1 *(i) If $A'_t \equiv S_t$ for all $t \in T$ (noncooperative situation), Assumption 2.2.5 holds automatically and Assumption 2.2.7(i) holds if $U'_t(s, \cdot, \cdot)$ is continuous on $\bar{\mathcal{S}}_\Sigma \times \mathbb{R}^m$ for every $s \in S_t$.*
(ii) In the terminology of [47, Proposition 1], Assumption 2.2.7(i) states that $U'_t(s, \bar{f}, y)$ is feasible path transfer lower semicontinuous in s with respect to A'_t *for every $t \in T$.*
(iii) Assumption 2.2.7(ii) holds if for every $t \in \bar{T}$ and $(\bar{f}, y) \in \bar{\mathcal{S}}_\Sigma \times \mathbb{R}^m$ the set $A'_t(\bar{f}, y)$ is convex and the function $U'_t(\cdot, \bar{f}, y)$ is quasiconcave on $A'_t(\bar{f}, y)$.

(iv) Assumptions 2.2.5(iii) and 2.2.6(ii) serve purely to guarantee $(\mathcal{T}\times\mathcal{B}(S))\cap\mathrm{gph}\,\Sigma$*-measurability of the graph of the multifunction* $t\mapsto \mathrm{argmax}_{s\in A'_t(\bar f,y)}U'_t(s,\bar f,y)$ *for every* $(\bar f,y)\in\bar{\mathcal{S}}_\Sigma\times\mathbb{R}^m$*.*
(v) As will become clear in the proof, the linearity of the action universe S*, as postulated in Assumption 2.2.2, is really only needed to obtain barycenters (pointwise) of mixed actions by players* $t\in\bar T$*. In other words, one could introduce two separate action universes, viz.* $\hat S$ *(for players* $t\in\bar T$*) and* $\bar S$ *(for players* $t\in\hat T$*). In such a setup only* $\bar S$ *would have to be as in Assumption 2.2.2, and* $\hat S$ *could be of the same type as in Assumption 2.1.1. In particular, this means that for the special case* $\bar T=\emptyset$*,* $\hat T:=T$ *in Theorem 2.2.1 we can replace Assumption 2.2.2 by Assumption 2.1.1.*
(vi) Another extension that is easy by the way purification is used in the proof of Theorem 2.2.1 is as follows. Instead of the externality component $\hat d$*, defined above, we could also have* $\hat d$ *equal to a countable sequence* $(\hat d_j)$*. This sequence would correspond to some countable measurable partitioning* $(\hat T_j)$ *of* $\hat T$*, and each* $\hat d_j$ *would have the same structure as* $\hat d$ *studied above, but relative to* $\hat T_j$ *instead of* $\hat T$*. Thus, for each* j *there would be* m_j *integrands* $g_1^j,\ldots,g_{m_j}^j$ *in* $\mathcal{G}_{\Sigma,C}(\hat T_j)$*, with* $\hat d_j(f):=(\int_{\hat T_j}g_i^j(t,f(t))\mu(dt))_{i=1}^{m_i}$*, and the new externality* $\hat d(f)$ *would now be* $(\hat d_j(f))_j$*.*

Theorem 2.2.1 (pure equilibrium existence result) *Under the Assumptions 2.2.1 to 2.2.7 there exists a* pure Cournot-Nash equilibrium *for the above pseudogame* Γ'*. That is, there exists a pure action profile* $f_*\in\mathcal{S}_\Sigma$ *such that*

$$f_*(t)\in\mathrm{argmax}_{s\in A'_t(d(f_*))}U'_t(s,d(f_*)) \text{ for } \mu\text{-a.e. } t \text{ in } T.$$

Observe that in the extreme case $\hat T=T$, with $(T,\mathcal{T},\mu)$ nonatomic, this result is entirely about purification by nonatomicity. In this capacity, for instance, it was shown in [11] to generalize the main result of [43, Theorem 2]; that result has a finite-dimensional action universe S, uses $g_i(t,s):=i$-th coordinate of s and works with integrable boundedness assumptions for Σ, as in Example 2.2.1. See, however, [37, Theorem 1] for a rather special equilibrium result by purification that is apparently not covered by Theorem 2.2.1. The above result, which will be proven later in subsection 3.2, is [15, Theorem 2.1], but with two additional improvements: (1) The current formulation of Assumption 2.2.7 means that certain continuity conditions that appear in [15, Assumptions 2.4, 2.5] can be replaced by mere upper semicontinuity conditions. (2) Assumption 2.2.5(iii) is less demanding than the corresponding part of [15, Assumption 2.4]. As explained in [15], Theorem 2.2.1 subsumes the extensions of Schmeidler's original result, obtained in [36, Theorems 7.1, 7.8, 7.11, 7.13] and [34, Theorem 4.7.3].

There is an obvious consistency question regarding the modeling of the payoffs in the pseudogames Γ and Γ'; it seems to have received only scant attention in the literature. Let us discuss this question only in terms of Γ'; a quite similar discussion can also be given for Γ. The point is that, any given action profile f completely specifies player t's action $f(t)$, which could affect the freedom of choice for the variable s in the payoff function $U'_t(s,d(f))$. In response, let us observe first that for players $t\in\hat T$ (these are "nonatomic players" by Assumption 2.2.1(ii)) the consistency issue does not arise: The action profile $f\,|_{\hat T}$ only influences the payoff $U'_t(s,d(f))$ via the aggregate $\hat d(f)=\hat d(f\,|_{\hat T})$ and this clearly does not determine the action $f(t)$ for any nonatomic player. However, for players t in $\bar T$ the response has to be more subtle, since $\bar d(f):=f\,|_{\bar T}$. For such players the model used in this paper still reflects proper modeling practice if $U'_t(s,d(f))$ is in addition supposed to be of a composite form, say $U'_t(s,d(f)):=U''_t(s,\pi_t(\bar d(f)),\hat d(f))$, where the mapping π_t is such that $\pi_t(\bar d(f))$ does *not* determine the value $f(t)$, i.e., player t's own action under f. So, rather than directly depending on $\bar d(f)$, the payoff depends on some "abstract" $\pi_t(\bar d(f))$ of $\bar d(f)$. As a concrete example, let us observe that in the original model of Schmeidler [45] (who, in the present terminology, works with $\hat T=T$), all such mappings π_t can be taken identically equal to the canonical L^1-space quotient mapping π. That is to say his "abstract" of $d(f)=f$ is simply the L_1-equivalence class $\pi(f)$ consisting of all functions that are a.e. equal to f. This choice reflects proper modeling, because knowledge of the equivalence class $\pi(f)$ does not specify anything about the action $f(t)$ taken by any particular player t under the profile f (recall that [45] works with $T=[0,1]$ and Lebesgue measure, so that each player is nonatomic). Much of the subsequent literature on continuum games has more or less adopted this model, although not always with the understanding that the measure space $(T, \mathcal{T}, \mu)$ is nonatomic.

In contrast, in games or pseudogames with at most countably many players the consistency question surfaces very keenly, because each player would be given positive μ-measure with $\mathcal{T} := 2^T$ – e.g., cf. [11, Theorem 3.1.1] (a quite similar situation arises if one considers additional "atomic players" in the above continuum game model). The standard formulations of such games simply realize consistency by working with $\pi_t(f) := f^{-t} := (f(\tau))_{\tau \neq t}$, etc. One might well wonder why such an effective device has not been used for games with a measure space of players. The reason is that in the continuum setting those same functions $\pi_t(f) := f^{-t}$ would suddenly present formidable technical complications, because of the fact that the joint evaluation map $(t, f) \mapsto f(t)$ need not be measurable in any standard way [30]. This fact has been overlooked in the strand of the continuum game literature that deals with models with unordered preferences *à la* Shafer-Sonnenschein, where, as a consequence, certain striking incompatibilities occur [19].

3 Proofs and equivalence

In this section we first prove Theorem 2.1.1 in subsection 3.1. Its proof is an application of Kakutani's fixed point theorem, which is topologically made possible by some of the most fundamental results of Young measure theory (these are recapitulated for the convenience of the reader). Recall that this theory centers around an extension of the classical narrow topology from probability measures to transition probabilities. Following this, Theorem 2.2.1 is proven in subsection 3.2, essentially by reformulating the existence problem of subsection 2.2 in terms of Theorem 2.1.1 and by adding some purification arguments. Finally, the equivalence of Theorems 2.1.1 and 2.2.1 is demonstrated in Proposition 3.3.1.

3.1 Proof of Theorem 2.1.1

In this subsection let us first recall the only three results about the narrow topology on $\mathcal{R}$ that we shall need in the proof of Theorem 2.1.1.

Proposition 3.1.1 *The narrow topology on $\mathcal{R}$ is semimetrizable.*

This result depends heavily on our initial assumption that the measure space $(T, \mathcal{T}, \mu)$ is separable. For a proof of the above result see [5, Proof of Lemma A.3], [18, Theorem 4.6] or [17, Theorem 4.5].

Theorem 3.1.1 *The subset $\mathcal{R}_\Sigma$ of $\mathcal{R}$ is narrowly compact.*

This follows directly from [6, Theorem 2.3], as extended from metrizable Lusin to metrizable Suslin spaces in [7] or, using Proposition 3.1.1 above, from [18, Theorem 4.10], by observing that, because of Assumption 2.1.2, setting $h(t, s) := 0$ if $s \in S_t$ and $h(t, s) := +\infty$ if $s \in S \backslash S_t$, defines an inf-compact normal integrand h (i.e., h belongs to the class $\mathcal{H}^{bb}(T; S)$ of [6, 7]). See [13, Corollary 2.2] and its proof for more details. The next result can be found in [5, Theorem I], [10, Appendix A], [18, Theorem 4.12] and in [17, Theorem 4.15].

Theorem 3.1.2 *If a sequence (η_n) converges narrowly to $\bar{\eta}$ in $\mathcal{R}$, then pointwise, for a.e t in T, the support* $\mathrm{supp}\ \bar{\eta}(t)$ *of the probability measure $\bar{\eta}(t)$ is contained in the set $\cap_{p=1}^{\infty} \mathrm{cl}\ \cup_{n \geq p} \mathrm{supp}\ \eta_n(t)$.*

This property expresses a kind of sequential upper semicontinuity of the (pointwise) supports; the set figuring in the above statement is called the *Painlevé-Kuratowski limes superior* and denoted as $\mathrm{Ls}_n \mathrm{supp}\ \eta_n(t)$.

Proof of Theorem 2.1.1. Evidently, $\delta_* \in \mathcal{R}_\Sigma$ is a mixed CNE if and only if $\delta_* \in F(\delta_*)$, where $F(\delta)$ stands for the set of all $\eta \in \mathcal{R}_\Sigma$ such that $\eta(t)(M_\delta(t)) = 1$ for μ-a.e. t in T. Here $M_\delta(t) := \mathrm{argmax}_{s \in A_t(\delta)} U_t(s, \delta))$. Therefore, the proof is entirely based on an application of Kakutani's fixed point theorem to $F : \mathcal{R}_\Sigma \to 2^{\mathcal{R}_\Sigma}$. Steps 1-2 below guarantee that $\mathcal{R}_\Sigma$ has the right compactness and convexity properties for such an application, and steps 3-5 show that F has the right semicontinuity properties. Step 6 applies Kakutani's theorem.

Step 1: compactness/convexity/nonemptiness of $\mathcal{R}_\Sigma$. By Theorem 3.1.1 the set $\mathcal{R}_\Sigma$ is compact for the narrow topology. Also, $\mathcal{R}_\Sigma$ is trivially convex in $\mathcal{R}$ and it was already seen before that $\mathcal{R}_\Sigma$ is nonempty.

Step 2: a vector space setting for $\mathcal{R}_\Sigma$. The intended application of Kakutani's theorem requires a topological vector space setting. Obviously, the classical narrow topology can be extended from $M_1^+(S)$ to the space $M(S)$ of all signed bounded measures on $(S, \mathcal{B}(S))$. Therefore, the vector space $\mathcal{M}$ spanned by $\mathcal{R}$ is the space of all functions from T into $M(S)$ that are measurable with respect to $\mathcal{T}$ and $\mathcal{B}(M(S))$. Equip $\mathcal{M}$ with the coarsest topology for which all functionals I_g : $\delta \mapsto \int_T [\int_S g(t,s)\delta(t)(ds)]\mu(dt)$, $g \in \mathcal{G}_C(T;S)$, are continuous (note that these functionals are well defined). When restricted to $\mathcal{R}$, this topology is the narrow topology that was defined previously.

Step 3: upper semicontinuity of $M_{\cdot}(t)$. By the Weierstrass theorem, $M_\delta(t)$ is a nonempty compact subset of S_t for every $t \in T$ and $\delta \in \mathcal{R}_\Sigma$ (use Assumptions 2.1.2(i), 2.1.4(i)). Moreover, $\delta \mapsto M_\delta(t)$ is upper semicontinuous for arbitrary $t \in T$. To see this, it is enough to prove that $M_{\cdot}(t)$ has the closed graph property (by compactness of S_t): So let $(s_n, \delta_n) \to (\bar{s}, \bar{\delta})$ with $s_n \in M_{\delta_n}(t)$ for every n, i.e., $s_n \in A_t(\delta_n)$ and $U_t(s_n, \delta_n) = \sup_{s \in A_t(\delta_n)} U_t(s, \delta_n)$. By Assumptions 2.1.4($i$) and 2.1.5 this identity leads to $U_t(\bar{s}, \bar{\delta}) \geq \sup_{s \in A_t(\bar{\delta})} U_t(s, \bar{\delta})$ in the limit. Also $\bar{s} \in A_t(\bar{\delta})$, because A_t has the closed graph property by Assumption 2.1.3(ii). So $\bar{s} \in M_{\bar{\delta}}(t)$, which proves the closed graph property of $M_{\cdot}(t)$.

Step 4: upper semicontinuity of F. Similar to step 3, it is enough to prove the closed graph property for F, because the values of F are contained in the compact set $\mathcal{R}_\Sigma$ (step 1). Here it is essential to convince oneself first that this classical result continues to hold on the semimetric space $\mathcal{R}_\Sigma$. To prove the closed graph property of F, let $(\eta_n, \delta_n) \to (\bar{\eta}, \bar{\delta})$ with $\eta_n \in F(\delta_n)$ for every n, i.e., $\eta_n(t)(M_{\delta_n}(t)) = 1$ for μ-a.e. t in T. This also means that for a.e. t in T and every n the support supp $\eta_n(t)$ of the probability measure $\eta_n(t)$ is contained in $M_{\delta_n}(t)$, for the latter set is closed by step 3. By Theorem 3.1.2, for a.e. t in T, this implies that supp $\bar{\eta}(t)$ is contained in the Painlevé-Kuratowski limes superior $\mathrm{Ls}_n M_{\delta_n}(t)$. By step 3, the latter set is contained in $M_{\bar{\delta}}(t)$, which finishes the proof.

Step 5: F has nonempty closed convex values. Fix $\delta \in \mathcal{R}_\Sigma$. The closedness of $F(\delta)$ follows *a fortiori* from the proof of the closed graph property of F in step 4. Convexity of $F(\delta)$ is trivial. Next, we prove nonemptiness of $F(\delta)$ by the application of a measurable selection theorem. To begin with, Assumptions 2.1.3(i) and 2.1.4(i) imply that the set $M_\delta(t)$ is nonempty for every $t \in T$ (by the Weierstrass theorem). Secondly. we show that M_δ has a measurable graph. Note that $s \in M_\delta(t)$ if and only if $s \in A_t(\delta)$ and $U_t(s, \delta) = \gamma_\delta(t)$, where $\gamma_\delta(t) := \sup_{s \in A_t(\delta)} \arctan U_t(s, \delta)$. By [25, III.39] the function γ_δ is $\mathcal{T}$-measurable (here completeness of $(T, \mathcal{T}, \mu)$ is used), so gph M_δ belongs to $\mathcal{T} \times \mathcal{B}(S)$ by Assumptions 2.1.3(iii) and 2.1.4(ii). It follows by the von Neumann-Aumann measurable selection theorem [25, III.22] that that there exists a measurable $f : T \mapsto S$ with $f(t) \in M_\delta(t)$ for every t in T. This implies that the Dirac Young measure ϵ_f (defined earlier) belongs to $F(\delta)$, which is thus seen to be nonempty.

Step 6: application of Kakutani's fixed point theorem. It is well-known that Ky Fan's original arguments in [32] do not require the Hausdorff space hypothesis [31, pp. 500-501]. In [15, Theorem A.2] this was used to obtain a non-Hausdorff version of Kakutani's theorem (in all other respect it is standard). Above, we saw that all properties needed for this fixed point result hold. So there exists $\delta_* \in \mathcal{R}_\Sigma$ with $\delta_* \in F(\delta_*)$, as desired.

3.2 Proof of Theorem 2.2.1

The foremost results needed in the derivation of Theorem 2.2.1 from Theorem 2.1.1 are as follows. The topological dual of S is denoted by S^*, and $< s, s^* > := s^*(s)$ indicates the usual duality.

Theorem 3.2.1 ([26, Proposition 26.3]) *If $K \subset S$ is nonempty compact and convex, then for every ν in $M_1^+(S)$ with* supp $\nu \subset K$ *there exists a unique $s_\nu \in K$ for which*

$$< s_\nu, s^* >= \int_K < s, s^* > d\nu \text{ for all } s^* \in S^*,$$

This unique element s_ν is denoted by bar ν.

Recall that s_ν is called the *barycenter* of the probability measure ν.

Corollary 3.2.1 *To every feasible mixed action profile $\delta \in \mathcal{R}_\Sigma$ there corresponds a pure action profile $f \in \mathcal{S}$ that satisfies $f(t) =$ bar $\delta(t)$ for a.e. t in T. This function f is essentially unique (i.e., but for null sets) and is denoted by* bar δ. *Moreover, its restriction $f\mid_{\bar{T}}$ belongs to the class $\bar{\mathcal{S}}_\Sigma$.*

Proof. By Theorem 3.2.1 the well-definedness of f follows from compactness in Assumption 2.2.3(i)-(ii). Admittedly, there may be an exceptional null set N of t's with $\delta(t)(S_t) \neq 1$, but for $t \in N$ one should set $f(t)$ equal to an arbitrary, fixed element of S. Observe that (bar $\delta)(t) =$ bar $(\delta(t))$ only for all t in $T \backslash N$. It follows that bar $\delta(t)$ belongs to S_t for a.e. t in $\bar{T}$, in view of Theorem 3.2.1 and Assumption 2.2.2(i). Measurability of f is seen as follows: For every $s^* \in S^*$ the above definition yields $< f(t), s^* >= \int_{S_t} < s, s^* > \delta(t)(ds)$ for all t in $T\backslash N$ and $< f(t), s^* >$ is constant on N. By [42, Proposition III.2.1] (here Assumption 2.2.3(iii) is used), one concludes that bar δ is measurable with respect to $\mathcal{T} \cap \bar{T}$ and $\mathcal{B}(S)$. This shows f to belong to $\bar{\mathcal{S}}$, because scalar and ordinary measurability of functions from T into S are equivalent, in view of Assumption 2.2.2 (namely, we can apply [46, Corollary 2, p. 101]). QED

Theorem 3.2.2 *The mapping $\delta \mapsto$ bar $\delta\mid_{\bar{T}}$ from $\mathcal{R}_\Sigma$ into $\bar{\mathcal{S}}_\Sigma$ is continuous with respect to the narrow and feeble topologies.*

Proof. Let $g \in \bar{\mathcal{G}}_{LC,\Sigma}$ and $\delta \in \mathcal{R}_\Sigma$ be arbitrary. Recall from subsection 2.2 that one has $g(t,\cdot) \in S^*$ for every $t \in \bar{T}$. So Theorem 3.2.1 and Corollary 3.2.1 give

$$g(t, \text{bar } \delta(t)) = \int_{S_t} g(t,s)\delta(t)(ds) = \int_S g(t,s)\delta(t)(ds) = \int_{S_t} g(t,s)\delta(t)(ds)$$

for a.e. t in $\bar{T}$. Integration over $\bar{T}$ therefore gives that $J_g(\text{bar } \delta) = I_{\tilde{g}}(\delta)$, where $\tilde{g}(t,s) := g(t,s)$ if $t \in \bar{T}$ and $s \in S_t$ and $\tilde{g}(t,s) = 0$ otherwise. Finally, note that $\tilde{g}$ belongs to the class $\mathcal{G}_{C,\Sigma}(T)$. QED

Theorem 3.2.3 (Lyapunov's theorem for Young measures) *If $\ell_1, \ldots, \ell_d : \hat{T} \times S \to \mathbb{R}$ are $(\mathcal{T} \times \mathcal{B}(S)) \cap \hat{T}$-measurable and if $\delta_0 \in \mathcal{R}$ is such that $\int_{\hat{T}}[\int_S |\ell_i(t,s)|\delta_0(t)(ds)]\mu(dt) < +\infty$ for all $1 \leq i \leq d$, then there exists a measurable function $f_0 : \hat{T} \to S$ such that $J_{\ell_i}(f_0) = I_{\ell_i}(\delta_0)$ for all i and $f_0(t) \in$* supp $\delta_0(t)$ *for a.e. t in $\hat{T}$.*

This is well-known in less general forms. The present version is [18, Theorem 5.3].

Proof of Theorem 2.2.1 Theorem 2.2.1 will be derived from Theorem 2.1.1 by the introduction of a mixed version Γ of the pseudogame Γ', which meets all conditions of Theorem 2.1.1. Thereupon, the mixed CNE action profile is transformed, both by barycentric arguments (on $\bar{T}$) and purification (on $\hat{T}$) into a pure CNE action profile.

Step 1: e and its continuity. Following [11], let us define a *mixed* externality mapping $e : \mathcal{R}_\Sigma \to \bar{\mathcal{S}}_\Sigma \times \mathbb{R}^m$ by setting $e := (\bar{e}, \hat{e})$ with

$$\bar{e}(\delta) := \text{bar } \delta\mid_{\bar{T}},\ \hat{e}(\delta) := (\int_{\hat{T}}\int_S [g_i(t,s)\delta(t)(ds)]\mu(dt))_{i=1}^m.$$

Observe that $\hat{e}(\delta) = (I_{\tilde{g}_i}(\delta))_{i=1}^m$, with $\tilde{g}_i \in \mathcal{G}_{C,\Sigma}$, where $\tilde{g}_i := g_i$ on gph $\Sigma \cap (\hat{T} \times S)$ and $\tilde{g}_i := 0$ on gph $\Sigma \cap (\bar{T} \times S)$. Thus, the function $\hat{e}$ is continuous by the facts about the narrow topology that were presented in subsection 2.1. By Theorem 3.2.2 $\bar{e}$, the other component of e, is also continuous.

Thus, e is continuous with respect to the narrow topology on $\mathcal{R}_\Sigma$ and the product of the feeble and Euclidean topologies on $\bar{\mathcal{S}}_\Sigma \times \mathbb{R}^m$.

Step 2: definition of a mixed pseudogame Γ. Define $A_t(\delta) := A'_t(e(\delta))$ and $U_t(s,\delta) := U'_t(s, e(\delta))$. Then all assumptions of Theorem 2.1.1 are easily seen to be met by the current Assumptions 2.2.2 to 2.2.7, in view of Step 1.

Step 3: application of Theorem 2.1.1. By Theorem 2.1.1 there exists $\delta_* \in \mathcal{R}_\Sigma$ such that $\delta_*(t)(\mathrm{argmax}_{s\in A'_t(e(\delta_*))} U'_t(s, e(\delta_*))) = 1$ for a.e. t in T.

Step 4: purification on $\hat{T}$. Theorem 3.2.3 can be applied in view of Assumption 2.2.1. So there exists a measurable function $\hat{f}_* : \hat{T} \to S$ such that $\hat{f}_*(t) \in \mathrm{supp}\ \delta_*(t) \subset \mathrm{argmax}_{s\in A'_t(e(\delta_*))} U'_t(s, e(\delta_*))$ a.e. on $\hat{T}$ and $\hat{d}(\hat{f}_*) = \hat{e}(\delta_*)$.

Step 5: construction of the pure CNE action profile. Set $f_*(t) := \mathrm{bar}\ \delta_*(t)$ on $\bar{T}$ and $f_*(t) := \hat{f}_*(t)$ on $\hat{T}$; then $e(\delta_*) = d(f_*)$. This gives $f_*(t) = \hat{f}_*(t) \in \mathrm{argmax}_{s\in A'_t(d(f_*))} U'_t(s, d(f_*))$ a.e. on $\hat{T}$. On $\bar{T}$ we can apply Theorem 3.2.1 to conclude that $f_*(t) := \mathrm{bar}\ \delta_*(t) \in \mathrm{argmax}_{s\in A'_t(d(f_*))} U'_t(s, d(f_*))$ a.e. on $\bar{T}$. This follows from the fact that the sets $\mathrm{argmax}_{s\in A'_t(d(f_*))} U'_t(s, d(f_*))$ are convex and compact for every $t \in \bar{T}$, in view of Assumptions 2.2.3(i), 2.2.6(i) and 2.2.7(ii). This finishes the proof. QED

3.3 Equivalence of Theorems 2.1.1 and 2.2.1

In the previous subsection Theorem 2.1.1 was shown to imply Theorem 2.2.1. In general, this is not an unusual implication. However, to find the converse implication would seem to be extremely rare (if not totally new), since the feasible mixed action spaces, i.e., the sets $M_1^+(S_t)$, $t \in T$, are usually stationed at a much higher level of generality than the action spaces S_t themselves.

Proposition 3.3.1 (equivalence) *Each of Theorems 2.1.1 and 2.2.1 implies the other result.*

Clearly, to prove this proposition it remains to derive Theorem 2.1.1 from Theorem 2.2.1. For this, it will be enough to make the special choice $\bar{T} = T$.

Step 1: definition of S *and* Σ'. Denote $S' := M(S)$, where $M(S)$ is as in subsection 3.1. In view of Assumption 2.1.1, S is a Suslin space for the classical narrow topology by [28, III.60] and [46, Theorem 3, p. 96]. Also, it is evident that S is locally convex by definition of the classical narrow topology. So Assumption 2.2.2 is met. Denote also $S'_t := \{\nu \in M(S) : \nu \in M_1^+(S) \text{ and } \nu(S_t) = 1\}$. In view of Assumption 2.1.2(i), S'_t is (classically) narrowly compact for every $t \in T$ [28, III.60], and it is trivially convex. By [25, Theorem IV.12] and Assumption 2.1.2(ii), the graph of $\Sigma' : t \mapsto S'_t$ is measurable. So Assumption 2.2.3 holds.

Step 2: $\mathcal{S}_{\Sigma'}$ *is* $\mathcal{R}_\Sigma$. By the above definition of Σ', it follows that $\mathcal{S}_{\Sigma'}$ is precisely the set $\mathcal{R}_\Sigma$ (recall from subsection 3.2 that scalar and ordinary measurability are the same for functions in $\mathcal{S}_{\Sigma'}$).

Step 3: feeble topology on $\mathcal{S}_{\Sigma'}$ *is narrow topology on* $\mathcal{R}_\Sigma$. Observe first that to every $g \in \mathcal{G}_C(T;S)$ there evidently corresponds $g' \in \bar{\mathcal{G}}_{LC,\Sigma'}$ via the formula $g'(t,\nu) := \int_S g(t,s)\,\nu(ds)$ (recall again that here $\bar{T} = T$). So all I_g, $g \in \mathcal{G}_C(T;S)$, are feebly continuous on $\mathcal{R}_\Sigma = \mathcal{S}_{\Sigma'}$. Conversely, let $g' \in \bar{\mathcal{G}}_{LC,\Sigma'}$ be arbitrary. By [26, Proposition 22.4] the topological dual $(S')^*$ of S is the set of all functionals $\nu \mapsto \int_S c\,d\nu$, $c \in \mathcal{C}_b(S)$, on $S' := M(S)$. Thus, by definition of $\bar{\mathcal{G}}_{LC,\Sigma'}$, for every $t \in T$ there exists $c_t \in \mathcal{C}_b(S)$ such that $g'(t,\nu) = \int_S c_t\,d\nu$ for all $\nu \in M(S)$. Observe that this gives $g'(t,\epsilon_s) = c_t(s) =: g(t,s)$ for all $t \in T$ and $s \in S$. By evident measurability of $(t,s) \mapsto (t,\epsilon_s)$, this implies that g is $\mathcal{T} \times \mathcal{B}(S)$-measurable. As in the previous case, the resulting formula is $g'(t,\nu) = \int_S g(t,s)\nu(ds)$. Finally, $g' \in \bar{\mathcal{G}}_{LC,\Sigma'}$ also implies that there exists $\phi_{g'} \in \mathcal{L}^1_{\mathbb{R}}(T,\mathcal{T},\mu)$ such that $\phi_{g'}(t) \geq \sup_{\nu\in S'_t} |g'(t,\nu)| = \sup_{s\in S_t} |g'(t,\epsilon_s)| = \sup_{s\in S_t} |g(t,s)|$ for every $t \in \bar{T}$. Hence, if one sets $\tilde{g}(t,s) := g(t,s)$ if $t \in \bar{T}$ and $s \in S_t$, and $\tilde{g}(t,s) := 0$ otherwise, then $\tilde{g}$ belongs to the class $\mathcal{G}_{C,\Sigma}$, defined in subsection 2.1, and $J_{g'}(\delta) = I_{\tilde{g}}(\delta)$ for every $\delta \in \mathcal{S}_{\Sigma'} = \mathcal{R}_\Sigma$. The conclusion is that the two topologies on $\mathcal{S}_{\Sigma'} = \mathcal{R}_\Sigma$ are the same.

Step 4: definition and properties of A'. Recall again that here $\bar{T} = T$, so that $\bar{\mathcal{S}}_\Sigma = \mathcal{R}_\Sigma = \mathcal{S}_{\Sigma'}$. For $\delta \in \mathcal{S}_{\Sigma'}$ one sets $A'_t(\delta) := \{\nu \in M_1^+(S) : \nu(A_t(\delta)) = 1\}$. Then Assumption 2.2.5($i$) holds by [28, III.58, III.60]. Also, in view of Assumption 2.1.3(ii), the corresponding Assumption 2.2.5(ii) holds by a well-known upper semicontinuity property *à la* Kuratowski (for convergence in the classical narrow topology of the supports of a sequence in $M_1^+(S)$ [10, Corollary A.2]. Note that this is the "classical" analogue of a similar property already used for mixed action profiles in the proof of Theorem 2.1.1 above.

Step 5: definition and properties of U'. For $\delta \in \mathcal{S}_{\Sigma'}$ one defines $U_t'(\nu,\delta) := \int_{S_t} \arctan U_t(s,\delta)\nu(ds)$ (the arctangent transformation serves to keep the integrand bounded, whence integrable). Then it is standard (apply [24, Theorem 3.2]) to show that U_t' is upper semicontinuous on $S_t' \times \mathcal{R}_\Sigma$, thanks to the fact that $\mathcal{R}_\Sigma$ is semimetrizable by Proposition 3.1.1). Also, it is standard to prove that $(t,\nu) \mapsto U_t'(\nu,\delta)$ is product measurable for every $\delta \in \mathcal{S}_{\Sigma'}$. So the conclusion is that Assumption 2.2.6 also holds.

Step 6: verification of Assumption 2.2.7. By the above definitions,

$$\sup_{\nu \in A_t'(\delta)} U_t'(\nu,\delta) = \sup_{s \in A_t(\delta)} \arctan U_t(s,\delta) = \arctan \sup_{s \in A_t(\delta)} U_t(s,\delta)$$

and the latter expression is clearly lower semicontinuous in δ by Assumption 2.1.5(i) and monotonicity/continuity of the arctangent function. Also, the above shows that the set $\text{argmax}_{\nu \in A_t'(\delta)} U_t'(\nu,\delta)$ is identical to the set $\{\nu \in M_1^+(S) : \nu(\text{argmax}_{s \in A_t(\delta)} U_t(s,\delta)) = 1\}$, which is trivially convex.

The proof is now virtually finished: Theorem 2.2.1 has been shown to apply, and, writing δ_* for f_*, this gives the existence of $\delta_* \in \mathcal{S}_{\Sigma'} = \mathcal{R}_\Sigma$ such that $\delta_*(t)(\text{argmax}_{s \in A_t(\delta_*)} U_t(s,\delta_*)) = 1$ for a.e. t in $\bar{T} = T$ (here the last part of Step 6 is used again).

3.4 Nonmeasurable versions

Let us very briefly consider two cases where measurability plays no role, either because T is at most countable and $\mathcal{T} = 2^T$ (call this case (i)), causing measurability of the profiles to be automatic, or because measurability of the profiles is no longer desired (call this case (ii)). In both cases the nonatomic part cannot figure; i.e., one has $\hat{T} := \emptyset$.

Case (i): Because the Suslin property is only instrumental for measurability with respect to $\mathcal{T}$, which is now automatic, the proofs of Theorems 2.1.1 and 2.2.1 show that one can remove the adjective "Suslin" from Assumptions 2.1.1 and 2.2.2, provided that one continues to suppose them metrizable (recall the introduction of ρ in subsection 2.1) or at least semimetrizable, as a closer inspection of the adapted proof shows. Also, there is no longer a need to keep all S_t contained in one and the same action universe S (but one could always re-create such inclusion by means of direct sums). Further, one can systematically replace "for μ-almost every t in T" by "for every $t \in T$", since one can work with, say, $\mu(\{t_i\}) := 2^{-i}$. Observe that now $\mathcal{R}_\Sigma = \Pi_{t \in T} M_1^+(S_t)$ and $\mathcal{S}_\Sigma = \Pi_{t \in T} S_t$, and on these Cartesian products the narrow and feeble topologies simply coincide with the usual product topologies (here the factors $M_(^+S_t)$, $t \in T$, are equipped with the classical narrow topology).

Case (ii): Unlike case (i), which is a special case of the general model considered here, new proofs have to be given of the counterparts of Theorems 2.1.1 and 2.2.1 that discard the measurability aspect. However, their *statements* take the form indicated in the previous case (i). Results of this kind are well-known and need not be repeated here; cf. [34, Theorem 4.7.2].

4 New applications

As mentioned before, the applications given in [11] and [15] are also applications of Theorems 2.1.1 and 2.2.1. Their details can be found in those papers. Here we shall continue two major lines from [11], viz. applications to existence of Cournot-Nash equilibrium distributions (subsection 4.1) and to existence of Cournot-Nash equilibria in more or less complicated games with a measure space of players (subsection 4.2). In addition, in subsection 4.4 new light is also shed on the connection of our model with games with incomplete information in the sense of Harsanyi.

4.1 CNE-distributions in anonymous games

In Theorem 4.1.1 below two separate results by Rath on the existence of CNE distributions in anonymous games *à la* Mas-Colell [39], namely Theorems 1 and 3 in [44], will be unified and generalized. The contribution made by [44] is in the line of [27]; it consists of specifying conditions that allow the payoff functions to be discontinuous (and more so than similar results of this kind, given in [9]). However, just as other existence results involving CNE distributions [9, 13], these results can be seen as a specialization of Theorem 2.1.1, that is, of a model for existence of CNE that goes considerably beyond the CNE distribution setting.

Recall that in anonymous games in the sense of Mas-Colell a player's type is made up entirely of his/her payoff function. This payoff function only depends on the (mixed) action profile via some marginal probability distribution generated by that profile on the action universe. The latter causes the anonymity feature. Let us now specify the following anonymous game Δ. As before, let S denote an action universe. The following repeats Assumption 2.1.1:

Assumption 4.1.1 *S is a completely regular Suslin space.*

Let T be a set of functions $t : S_t \times M_1^+(S) \to \mathbb{R}$, where the factor $S_t \subset S$ determines the domain of definition $S_t \times M_1^+(S)$ of the function t. As before, the notation $\Sigma : t \mapsto S_t$ is used frequently and T is equipped with a σ-algebra $\mathcal{T}$ and a measure μ, which is now also supposed to be a *probability* measure. The following repeats Assumption 2.1.2:

Assumption 4.1.2 (*i*) *For every $t \in T$ the set $S_t \subset S$ is nonempty and compact.*
(*ii*) $\mathrm{gph}\ \Sigma \in \mathcal{T} \times \mathcal{B}(S)$.

Below the set of probability measures $M_1^+(S)$ is equipped with the classical narrow topology [24, 28, 46].

Assumption 4.1.3 (*i*) *For every $t \in T$ the function $t : S_t \times M_1^+(S) \to \mathbb{R}$ is upper semicontinuous and such that*

$$\nu \mapsto \sup_{s \in S_t} t(s, \nu) \text{ is lower semicontinuous on } M_1^+(S).$$

(*ii*) *For every $\nu \in M_1^+(S)$ the function $(t, s) \mapsto t(s, \nu)$ is $(\mathcal{T} \times \mathcal{B}(S)) \cap \mathrm{gph}\ \Sigma$-measurable.*

Theorem 4.1.1 *Under the above Assumptions 4.1.1 to 4.1.3 there exists a* Cournot-Nash equilibrium distribution *for the anonymous game Δ, That is, there exists a probability measure p_* on $T \times S$ such that*

$$p_*(\cdot \times S) = \mu \text{ and } p_*(\{(t, s) \in \mathrm{gph}\ \Sigma : s \in \mathrm{argmax}_{s' \in S_t} t(s, p_*(T \times \cdot))\}) = 1.$$

Proof. Let us apply Theorem 2.1.1 by making the following substitutions: set $A_t(\delta) \equiv S_t$, as in (2.1), and $U_t(s, \delta) := t(s, \nu_\delta)$, where $\nu_\delta \in M_1^+(S)$ is defined by

$$\nu_\delta(B) := \int_T \delta(t)(B)\mu(dt),\ B \in \mathcal{B}(S)$$

(recall that $\mu(T) = 1$). Now the mapping $\delta \mapsto \nu_\delta$ is continuous from $\mathcal{R}$, equipped with the narrow topology, into $M_1^+(S)$, equipped with the classical narrow topology. This follows directly from the fact that $\int_S c\, d\nu_\delta = I_g(\delta)$ for any $c \in \mathcal{C}_b(S)$, where $g(t, s) := c(s)$ defines a Carathéodory integrand on $T \times S$. Hence, it is evident that Assumption 2.1.4(i) holds, and Assumption 2.1.4(ii) follows of course by Assumption 4.1.3(ii). Also, Assumption 2.1.5 holds by $\sup_{s \in S_t} t(s, \nu_\delta)$, in view of the above continuity result and Assumption 4.1.3(i). Finally, as observed in Remark 2.1.1(i), Assumption 2.1.3 holds here automatically. So the theorem can be applied, which gives the existence of $\delta_* \in \mathcal{R}_\Sigma$ with the properties as stated in Theorem 2.1.1. Now the canonical product measure that is generated by the "starting" probability measure μ and the transition probability δ_* [42, III.2], is immediately seen to form the desired equilibrium distribution p_*. QED

The structural analogy of the above proof with the proof of Theorem 2.2.1 is worth noting: apart from the separate purification on $\hat{T}$, in the latter proof it was the continuity of $\delta \mapsto$ bar $\delta \mid_{\hat{T}}$ that immediately caused the result of Theorem 2.1.1 to apply. Here this continuity is replaced by the continuity of the special externality mapping $\delta \mapsto \nu_\delta$ that characterizes the Mas-Colell model. Theorem 4.1.1 generalizes both Theorem 1 and 3 of Rath [44] (in turn, Rath's Theorem 1 generalizes the original result by Mas-Colell in [39]). Let us see why this is so. In [44] one has S compact metric and $S_t \equiv S$; this obviously meets the Assumptions 4.1.1 and 4.1.2. Let $\mathcal{P}$ be the set of *all* bounded upper semicontinuous functions $t : S \times M_1^+(S) \to \mathbb{R}$ such that $\nu \mapsto \sup_{s \in S} t(s, \nu)$ is lower semicontinuous on $M_1^+(S)$. In [44] the set $\mathcal{P}$ is endowed with a probability measure μ on $(\mathcal{P}, \mathcal{B}(\mathcal{P}))$; here the Borel σ-algebra is taken with respect to either the usual supremum norm topology (Theorem 1) or the hypotopology (Theorem 2). In the first situation $\mathcal{P}$ is denoted as $\mathcal{P}^S$ in [44], and in the second situation as $\mathcal{P}^H$. Now observe that each of the above choices of topology causes the mapping $(t, s) \mapsto t(s, \nu)$ to be upper semicontinuous on $\mathcal{P} \times S$, whence $\mathcal{B}(\mathcal{P} \times S)$-measurable, for every fixed $\nu \in M_+^1(S)$.

First, let us bring Theorem 4.1.1 to bear on Theorem 1 of [44]. In that result μ is *tight*, so there exists a sequence of compacts $K_n \subset \mathcal{P}$ with $\mu(\mathcal{P} \backslash T) = 0$ for $T := \cup_{n=1}^\infty K_n$. Let us also set $\mathcal{T} := \mathcal{B}(T)$, then $\mathcal{T} = \mathcal{B}(\mathcal{P}) \cap T$. Since $\mathcal{P} = \mathcal{P}^S$ is equipped with a metric (supremum norm), it follows that T is separable, whence second countable, so the restriction of the Borel σ-algebra $\mathcal{B}(\mathcal{P} \times S)$ to $T \times S$ is equal to the restriction of $\mathcal{B}(\mathcal{P}) \times \mathcal{B}(S)$ to that same set. In view of the preceding, this shows that the restriction of $(t, s) \mapsto t(s, \nu)$ to $T \times S$ is measurable with respect to $\mathcal{T} \times \mathcal{B}(S)$. Hence, also the remaining Assumption 4.1.3(ii) of Theorem 4.1.1 is met. Clearly, by $\mu(\mathcal{P} \backslash T) = 0$, one can add the remaining functions in $\mathcal{P} \backslash T$ to the statement resulting from Theorem 4.1.1 to regain Theorem 1 of [44].

Next, let us obtain Theorem 3 of [44]. In this case the choice $T := \mathcal{P}$ suffices, since $\mathcal{P} = \mathcal{P}^H$ is well-known to be separable and metrizable for the hypotopology (cf. Propositions 8,9 in [44]). So satisfaction of the remaining Assumption 4.1.3(ii) of Theorem 4.1.1 follows as above.

Note from the above two derivations that, given Rath's compactness and semicontinuity conditions, the application of Theorem 4.1.1 – or, in the background, Theorem 2.1.1 – is really reduced to a measurability question (i.e., the identity of certain σ-algebras).

4.2 Existence of BNE for games with differential information

We generalize the main Bayesian-Nash equilibrium existence result of Kim and Yannelis in [38]. Consider the following Bayesian game $\tilde{\Gamma}$. Let S be an action universe. The following assumption replicates Assumption 2.2.2.

Assumption 4.2.1 *S is a Suslin locally convex topological vector space.*

By an earlier remark, made in subsection 2.1, this is only more general in *appearance* than the assumption that S be metrizable Suslin, in view of the other assumptions below. Let $(\Omega, \mathcal{F}, P)$ be a probability space of *states of nature* and, as before, let $(T, \mathcal{T}, \mu)$ be a measure space of players. Every player $t \in T$ obtains information about the realized state of nature via his/her *informational σ-algebra* $\mathcal{F}_t$, which is a given sub-σ-algebra of $\mathcal{F}$ (differential information). As in [38], the following drastic assumption is unavoidable for technical reasons; these are mainly of a topological nature – cf. Proposition 4.3.2.

Assumption 4.2.2 *The set Ω is at most countable.*

Rather than relabeling the atoms of $\mathcal{F}$, we shall assume without loss of generality that $\mathcal{F}$ is the power set 2^Ω. Let $\tilde{\Sigma} : T \times \Omega \to 2^S$ be a given multifunction. For each player $t \in T$ his/her *feasible* actions constitute the subset $\tilde{\Sigma}(t, \omega)$, given that $\omega \in \Omega$ is the realized state of nature. Observe from part (ii) of the following assumption that this feasibility restriction is in accordance with player t's informational σ-algebra.

Assumption 4.2.3 (i) *For every $(t, \omega) \in T \times \Omega$ the set $\tilde{\Sigma}(t, \omega) \subset S$ is nonempty, convex and compact.*

(*ii*) gph $\tilde{\Sigma}(t,\cdot) \in \mathcal{F}_t \times \mathcal{B}(S)$ *for every* $t \in T$.
(*iii*) gph $\tilde{\Sigma} \in \mathcal{T} \times \mathcal{F} \times \mathcal{B}(S)$.

Let us write $\tilde{\Sigma}_\omega(t) := \tilde{\Sigma}(t,\omega)$ and denote the corresponding multifunctions by $\tilde{\Sigma}_\omega$, $\omega \in \Omega$. Then parts (*i*) and (*iii*) of the above assumption imply that these multifunctions have nonempty compact convex values and a $\mathcal{T} \times \mathcal{B}(S)$-measurable graph. Thus, for each $\omega \in \Omega$ the set $\mathcal{S}_{\tilde{\Sigma}_\omega}$ is defined in complete analogy to the set $\mathcal{S}_\Sigma$ of subsection 2.2. Hence, from now on each such space can be considered to be equipped with the (i.e., its own) feeble topology.

A *feasible pure action profile* is a function $\tilde{f} : T \times \Omega \to S$ that is measurable with respect to $\mathcal{T} \times \mathcal{F}$ and $\mathcal{B}(S)$, with $\tilde{f}(t,\cdot)$ being $\mathcal{F}_t$-measurable for every $t \in T$ and with $\tilde{f}(t,\omega) \in \tilde{\Sigma}(t,\omega)$ for μ-a.e. t in T and P-a.e. ω in Ω. Note that this means that for every player $t \in T$ the description $\tilde{f}(t,\cdot)$ of what player t could/should do under the various states of nature, takes into account the way in which player t processes information about that state (i.e., by way of $\mathcal{F}_t$-measurability). The set of all such feasible pure action profiles is denoted by $\mathcal{S}_{\tilde{\Sigma}}$. Note that for every $\omega \in \Omega$ and $\tilde{f} \in \mathcal{S}_{\tilde{\Sigma}}$ the ω-section $\tilde{f}(\cdot,\omega)$ of $\tilde{f}$ belongs to $\mathcal{S}_{\tilde{\Sigma}_\omega}$.

The Bayesian nature of the model is reflected by the fact that each player $t \in T$ possesses a *Bayesian prior distribution*; this is a transition probability π_t which expresses player t's *interim beliefs* about the actually realized state of nature, that is to say, beliefs formulated after having gained him/herself information (i.e., partially, via $\mathcal{F}_t$) about it.

Assumption 4.2.4 (*i*) *For every* $t \in T$ π_t *is a transition probability with respect to* $(\Omega, \mathcal{F}_t)$ *and* $(\Omega, \mathcal{F})$, *i.e., for every* $A \in \mathcal{F}$ *the function* $\pi_t(\cdot)(A)$ *is* $\mathcal{F}_t$*-measurable.*
(*ii*) *For every* $A \in \mathcal{F}$ *the function* $(t,\omega) \mapsto \pi_t(\omega)(A)$ *is* $\mathcal{T} \times \mathcal{F}$*-measurable.*

For every $(t,\omega) \in T \times \Omega$ let $u_{t,\omega} : Z_t \times \mathcal{S}_{\tilde{\Sigma}_\omega} \to \mathbb{R}$ be a given *utility function*. Here $Z_t := \cup_{\omega\in\Omega}\tilde{\Sigma}(t,\omega)$. If player t in T were to know the realized state $\omega \in \Omega$ completely, he/she would assign utility value $u_{t,\omega}(s, \tilde{f}(\cdot,\omega))$ to his/her own action $s \in \tilde{\Sigma}(t,\omega)$ in the face of the action profile $\tilde{f} \in \mathcal{S}_{\tilde{\Sigma}}$. Shortly, we shall see how, using his/her prior distribution $\pi_t(\omega)$ as a Bayesian assessment of the realized state of nature, player t can convert this into an appraisal that is in line with his/her informational sub-σ-algebra $\mathcal{F}_t$.

Assumption 4.2.5 (*i*) *For every* $(t,\omega) \in T \times \Omega$ *the function* $u_{t,\omega} : Z_t \times \mathcal{S}_{\tilde{\Sigma}_\omega} \to \mathbb{R}$ *is upper semi-continuous.*
(*ii*) *For every* $(t,\omega) \in T \times \Omega$ *and* $s \in Z_t$ *the function* $u_{t,\omega}(s,\cdot) : \mathcal{S}_{\tilde{\Sigma}_\omega} \to \mathbb{R}$ *is continuous.*
(*iii*) *For every* $\omega \in \Omega$ *and* $f \in \mathcal{S}_{\tilde{\Sigma}_\omega}$ *the function* $(t,s) \mapsto u_{t,\omega}(s,f)$ *is* $(\mathcal{T} \times \mathcal{B}(S)) \cap$gph $Z_\cdot$*-measurable.*
(*iv*) *For every* $(t,\omega) \in T \times \Omega$ *there exists* $\phi_t \in \mathcal{L}^1_{\mathbb{R}}(\Omega, \mathcal{F}, \pi_t(\omega))$ *such that for every* $\omega' \in \Omega$

$$\sup_{s\in\tilde{\Sigma}(t,\omega'), f\in\mathcal{S}_{\tilde{\Sigma}_{\omega'}}} |u_{t,\omega'}(s,f)| \le \phi_t(\omega').$$

In part (*iii*) above gph $Z_\cdot$ refers to the graph of the multifunction $t \mapsto Z_t := \cup_{\omega\in\Omega}\tilde{\Sigma}(t,\omega)$, i.e., the $\mathcal{T} \times \mathcal{B}(S)$ measurable countable union $\cup_{\omega\in\Omega}$gph $\tilde{\Sigma}_\omega$.

Assumption 4.2.6 *For every* $(t,\omega) \in T \times \Omega$ *and* $f \in \mathcal{S}_{\tilde{\Sigma}_\omega}$ *the function* $u_{t,\omega}(\cdot, f)$ *is concave.*

Following [38], let us introduce the following Bayesian object to overcome the informational limitations inherent to the utility evaluation $(t,\omega,s,\tilde{f}) \mapsto u_{t,\omega}(s,\tilde{f}(\cdot,\omega))$, as mentioned above. Given the state $\omega \in \Omega$, player t's *conditional expected interim utility function* $U_{t,\omega} : \tilde{\Sigma}(t,\omega) \times \mathcal{S}_{\tilde{\Sigma}} \to \mathbb{R}$ is defined as follows:

$$U_{t,\omega}(s,\tilde{f}) := \int_\Omega u_{t,\omega'}(s,\tilde{f}(\cdot,\omega'))\pi_t(\omega)(d\omega'). \tag{4.1}$$

Existence of this integral (actually, by Assumption 4.2.2 it is an at most countable sum) is elementary, in view of Assumption 4.2.5(*iv*) and the fact that $\pi_t(\omega)$ is a probability on $\mathcal{F} = 2^\Omega$.

Theorem 4.2.1 *Under Assumptions 4.2.1 to 4.2.6 there exists a Bayesian Nash equilibrium action profile for the game $\tilde{\Gamma}$, i.e., there exists $\tilde{f}_* \in \mathcal{S}_{\tilde{\Sigma}}$ such that*

$$\tilde{f}_*(t,\omega) \in \mathrm{argmax}_{s \in \tilde{\Sigma}(t,\omega)} U_{t,\omega}(s, \tilde{f}_*) \text{ for } \mu\text{-a.e. } t \text{ in } T \text{ and } P\text{-a.e. } \omega \text{ in } \Omega.$$

This result improves and generalizes the main result Theorem 5.2 of Kim and Yannelis [38] in several respects. They need S to be a separable Banach space (equipped with its weak topology), and their multifunction $\tilde{\Sigma}$ is, in addition to our conditions, also integrably bounded. This causes their counterpart of our $\mathcal{S}_\Sigma$ to be in an L^1-context (this is a very complicated quotient context, borrowed from [22]). Hence, on their counterpart of $\mathcal{S}_\Sigma$ they can work with the weak topology $\sigma(L^1, L^\infty)$, which the feeble topology used here generalizes (see Example 2.2.1). The remaining comparisons, which are all in favor of the above set of conditions, are left to the reader. We must only point out that Assumption B.1(ii) of [38] (which would constitute an improvement over our Assmuption 4.2.5) appears to be wrong, since such an assumption of strong-weak continuity of $u_{t,\omega}$ does not by itself imply joint continuity for the weak topology (which is the topology they work with), not even when $u_{t,\omega}(s,f)$ is concave in s. For instance, with their $T \times \Omega$ a singleton, consider the fact that the inner product mapping $(x,y) \mapsto \sum_{i=1}^\infty x_i y_i$ from the product of the unit ball in ℓ_2 with itself into the reals is strong-weak continuous, bilinear, but not jointly weakly continuous.

4.3 Proof of Theorem 4.2.1

Let us prepare for an application of Theorem 2.2.1, of course with $\bar{T} = T$ and with noncooperativity in force (i.e., $A_{t,\omega} \equiv \tilde{\Sigma}(t,\omega)$).

To begin with, let $\tilde{\mathcal{T}} \subset 2^{T \times \Omega}$ be the collection of all $E \in \mathcal{T} \times \mathcal{F}$ such that for every $t \in T$ the t-section $E_t := \{\omega \in \Omega : (t,\omega) \in E\}$ belongs to $\mathcal{F}_t$. Observe that $\tilde{\mathcal{T}}$ defines a σ-algebra on $\tilde{T} := T \times \Omega$. It is called the *progressive* σ-algebra in stochastics. For further coherence, let us denote $\tilde{\mu} := \mu \times P$ for the product measure on $(\tilde{T}, \tilde{\mathcal{T}})$). The measure space $(\tilde{T}, \tilde{\mathcal{T}}, \tilde{\mu})$ will now take the place of $(T, \mathcal{T}, \mu)$ as used in Theorem 2.2.1. Hereupon, observe that Assumption 4.2.3(ii)-(iii) amounts precisely to having gph $\tilde{\Sigma} \in \tilde{\mathcal{T}} \times \mathcal{B}(S)$. So, together with Assumption 4.2.3(i), this means that Assumption 2.2.2 has been met. In view of our adoption of noncooperativity, Assumption 2.2.5 is met vacuously (Remark 2.2.1(i)).

Denote $\tilde{\pi}(t,\omega)(A) := \pi_t(\omega)(A)$. Then, by definition of the progessive σ-algebra, Assumption 4.2.4 states precisely that $\tilde{\pi}$ is a transition probability with respect to $(\tilde{T}, \tilde{\mathcal{T}})$ and $(\Omega, \mathcal{F})$. Fix $\tilde{f} \in \mathcal{S}_{\tilde{\Sigma}}$. By Assumption 4.2.5($iii$) and $\mathcal{F} = 2^\Omega$ (i.e., Assumption 4.2.2) the function $(t,s) \mapsto u_{t,\omega'}(s, \tilde{f}(\cdot,\omega'))$ of (4.1) is $(\mathcal{T} \times \mathcal{B}(S)) \cap$gph Z-measurable for every ω' in the countable set Ω'. So the above progressive measurability property of $\tilde{\pi}$ implies that $(t,\omega,s) \mapsto U_{t,\omega}(s,\tilde{f})$ is $(\tilde{\mathcal{T}} \times \mathcal{B}(S)) \cap$ gph $\tilde{\Sigma}$-measurable. Hence, Assumption 2.2.6(ii) has been met.

We now use an analogue of Proposition 3.1.1 that can be proven just as easily, thanks to the separability of $(T, \mathcal{T}, \mu)$ and the fact that a countable subset of S' separates the points of S (by [25, III.32]). See [20] for details. We first phrase the result in the original context of subsection 2.2:

Proposition 4.3.1 *The feeble topology on $\mathcal{S}_\Sigma$ is semimetrizable.*

Of course, this proposition means that $\mathcal{S}_{\tilde{\Sigma}}$ is semimetrizable for its feeble topology (note that $(\tilde{T}, \tilde{\mathcal{T}}, \tilde{\mu})$ is also separable). This allows us to use only sequential arguments to verify continuity/semicontinuity in what follows.

We also need the following feeble to feeble continuity property of the ω-section mapping, which again draws heavily on Assumption 4.2.2:

Proposition 4.3.2 *For every $\omega \in \Omega$ the mapping $\tilde{f} \mapsto \tilde{f}(\cdot,\omega)$ from $\mathcal{S}_{\tilde{\Sigma}}$, equipped with the feeble topology, into $\mathcal{S}_{\tilde{\Sigma}_\omega}$, also equipped with its own feeble topology, is continuous.*

Proof. Fix any $\omega \in \Omega$. Given an arbitrary $g \in \mathcal{G}_{LC,\tilde{\Sigma}_\omega}$, we define $\tilde{g}(t,\omega',s) := g(t,s)$ if $\omega' = \omega$ and $\tilde{g}(t,\omega',s) := 0$ if $\omega' \neq \omega$. Then $\tilde{g}$ is easily seen to belong to $\mathcal{G}_{LC,\tilde{\Sigma}}$. Since

$$\int_{\tilde{T}} \tilde{g}(t,\omega',\tilde{f}(t,\omega'))\tilde{\mu}(d(t,\omega')) = P(\{\omega\}) \int_T g(t,\tilde{f}(t,\omega))\mu(dt),$$

the result follows by definition of the respective feeble topologies. QED

Using these two results, it is now easy to see that for every $(t,\omega) \in \tilde{T}$ the function $U_{t,\omega}$ is upper semicontinuous on $\tilde{\Sigma}(t,\omega) \times \mathcal{S}_{\tilde{\Sigma}}$ by an application of Fatou's lemma. Here Assumption 4.2.5(iv) provides integrable boundedness from above for the sequence, and Assumption 4.2.5(i) should be combined with Proposition 4.3.2. Conversely, in view of Assumption 4.2.5(ii) a similar application of Fatou's lemma (or – which has the same effect – by Lebesgue's dominated convergence theorem) gives that for every $(t,\omega) \in \tilde{T}$ and $s \in \tilde{\Sigma}(t,\omega)$ the function $U_{t,\omega}(s,\cdot)$ is continuous on $\mathcal{S}_{\tilde{\Sigma}}$. So Assumption 2.2.7$(i)$ holds by Remark 2.2.1(i). Finally, the integration operation in (4.1) obviously preserves the concavity, as guaranteed by Assumption 4.2.6. So Assumption 2.2.7(ii) holds by Remark 2.2.1(iii). We conclude that all assumptions of Theorem 2.2.1 have been shown to hold, since the model of Kim and Yannelis has been reduced to the one used in subsection 2.2. Application of Theorem 2.2.1 immediately implies that Theorem 4.2.1 holds.

4.4 Existence of BNE in games with private information

Let us show how the extensions of [6, 21] of the BNE existence result of Milgrom-Weber [41] (which is in mixed action profiles) follow from Theorem 2.2.1 (which is in pure action profiles!). This approach would seem to be somewhat in the spirit of [40]. Consider the following Bayesian game $\hat{\Gamma}$ *à la* Harsanyi [33].

Assumption 4.4.1 *The set T is at most countable.*

Assumption 4.4.2 *For every $t \in T$ the set S_t is a nonempty metrizable compact set.*

For every $t \in T$ let $(\Omega_t, \mathcal{F}_t)$ be a measurable space forming player t's space of *private observations*. Let P be a probability measure on the countable product space $(\Omega, \mathcal{F}) := \Pi_{t\in T}(\Omega_t, \mathcal{F}_t)$. The realizations in Ω are governed by P, but player $t \in T$ is only informed of his/her marginal outcome on Ω_t ("private information"). Clearly, this marginal outcome is governed by P_t, the marginal of P on the t-th factor space; i.e., $P_t(B) := P(\Pi_{\tau\in T,\tau\neq t}\Omega_t \times B)$. The following condition was also used in [41, 6, 21]:

Assumption 4.4.3 *P is absolutely continuous with respect to the product measure $\Pi_{t\in T}P_t$.*

For each $t \in T$ let $\mathcal{R}_t$ be the space of all transition probabilities with respect to $(\Omega_t, \mathcal{F}_t)$ and $(S_t, \mathcal{B}(S_t))$; this space is equipped with the narrow topology, introduced in subsection 2.1. Clearly, in using $\delta_t \in \mathcal{R}_t$, player $t \in T$ keeps to his/her allowed private information restriction, and uses mixed actions in $M_1^+(S_t)$ (it would be possible to introduce ω_t-dependency of the feasible mixed actions in the usual way, but this will not be done to keep the presentation simple). Let $S := \Pi_{t\in T}S_t$. We also use $S^{-t} := \Pi_{\tau\in T,\tau\neq t}S_\tau$. Each player $t \in T$ has a payoff function $u_t : \Omega \times S \to \mathbb{R}$, of which the following is required.

Assumption 4.4.4 (i) *For every $(t,\omega) \in T \times \Omega$ the function $u_t(\omega,\cdot) : S \to \mathbb{R}$ is continuous.*
(ii) *For every $t \in T$ the function $u_t : \Omega \times S \to \mathbb{R}$ is $\mathcal{F} \times \mathcal{B}(S)$-measurable.*
(iii) *For every $t \in T$ there exists $\phi_t \in \mathcal{L}^1_{\mathbb{R}}(\Omega, \mathcal{F}, P)$ such that for every $\omega \in \Omega$*

$$\sup_{s\in S} |u_t(\omega, s)| \leq \phi_t(\omega).$$

These assumptions allow us to introduce the following *expected payoff* functions $V_t : \Pi_{\tau\in T}\mathcal{R}_\tau \to \mathbb{R}$:

$$V_t((\delta_\tau)_{\tau\in T}) := \int_\Omega \int_S u_t(\omega, s)[\otimes_{\tau\in T}\delta_\tau](\omega)(ds)P(d\omega),$$

where $\otimes_{\tau\in T}\delta_\tau$ is the transition probability with respect to $(\Omega, \mathcal{F})$ and $(S, \mathcal{B}(S))$, defined by

$$[\otimes_{\tau\in T}\delta_\tau](\omega) := \Pi_{\tau\in T}\delta_\tau(\omega_\tau)$$

for $\omega := (\omega_\tau)_{\tau\in T}$ (i.e., one takes pointwise product measures).

Theorem 4.4.1 *Under Assumptions 4.4.3 to 4.4.4 there exists a Bayesian Nash equilibrium action profile for the game $\hat{\Gamma}$, i.e., there exists $\delta_* := (\delta_{*t})_{t\in T} \in \Pi_{t\in T}\mathcal{R}_t$ such that*[1]

$$\delta_{*t} \in \mathrm{argmax}_{\delta_t \in \mathcal{R}_t} V_t(\delta_t, \delta_*^{-t}) \textit{ for every } t \in T..$$

Proof. *Step 1: separable case.* Suppose in addition that for every $t \in T$ the σ-algebra $\mathcal{F}_t$ is countably generated (in particular, it is then separable). By Proposition 4.3.1, this implies that every $\mathcal{R}_t$ is semimetrizable for the topology of narrow convergence of transition probabilities. By Assumption 4.4.1, we can simplify the application of Theorem 2.2.1 as discussed in subsection 3.4. In particular, this means that a common action universe is not *per se* a requirement for the feasible action sets, and also that those action sets are allowed to be semimetrizable. For $t \in T$ set $\hat{\Sigma}(t) := \hat{S}_t$, with $\hat{S}_t$ defined to be the quotient of $\mathcal{R}_t$ for the obvious equivalence relation "equality P_t-a.e."; then by the above $\hat{S}_t$ is metrizable. By Theorem 3.1.1 $\hat{S}_t$ is compact for every $t \in T$; also, it is trivially convex. So Assumption 2.2.3 has been shown to hold. Observe that $\mathcal{S}_{\hat{\Sigma}} = \Pi_{t\in T}\hat{S}_t$ by Assumption 4.4.1; as already mentioned in subsection 3.4, the feeble topology on $\mathcal{S}_{\hat{\Sigma}}$ now coincides with the product topology. We define $U_t : \hat{S}_t \times \mathcal{S}_{\hat{\Sigma}} \to \mathbb{R}$ by $U_t(\delta, \hat{f}) := V_t(\delta, \hat{f}^{-t})$. Here we adopt standard notation that is explained in subsection 2.2 and the previous footnote. In addition, we abuse the notation a little – in the accepted way – in connection with the quotient setting in which we actually work (note, for instance, that a quotient counterpart of V_t should be defined on $\Pi_{\tau\in T}\hat{S}_\tau$ in an evident manner). Let us equip $(\Omega, \mathcal{F})$ with the measure $\Pi_{t\in T}P_t$, and let $\mathcal{R}_0$ be the set of all transition probabilities with respect to $(\Omega, \mathcal{F})$ and $(S, \mathcal{B}(S))$. By well-known facts about the tensor product of transition probabilities (see [6, Theorem 2.5] and [21, p. 389]), the mapping $(\delta_t)_{t\in T} \mapsto \otimes_{t\in T}\delta_t$, defined from $\mathcal{S}_{\hat{\Sigma}} = \Pi_{t\in T}\mathcal{R}_t$ into $\mathcal{R}_0$, is continuous with respect to the narrow topology on the latter space. Here Assumption 4.4.3 and the choice of $\Pi_t P_t$ as the leading measure on Ω play an important role. Fix any $t \in T$. Define $v_t(\omega, s) := u_t(\omega, s)r(\omega)$, where r is any fixed version of the Radon-Nikodym density of P with respect to $\Pi_{\tau\in T}P_\tau$. Assumption 4.4.4 causes $-v_t$ to belong to the class $\mathcal{G}_C(\Omega; S)$ of Carathéodory integrands with respect to the measure $\Pi_{\tau\in T}P_\tau$ on Ω. Hence, I_{v_t} is narrowly continuous on $\mathcal{R}_0$. Hence, by the continuity of the tensor product, observed above, and the obvious identity $V_t((\delta_\tau)_{\tau\in T}) = I_{v_t}(\otimes_{\tau\in T}\delta_\tau)$, the function U_t is upper semicontinuous on $\hat{S}_t \times \mathcal{S}_{\hat{\Sigma}}$. Hence, Assumption 2.2.6 is met (note that measurability in the variable t is trivial here), and also Assumption 2.2.7(i) (invoke Remark 2.2.1(i)). Finally, Assumption 2.2.7(ii) holds by the obvious affinity of $U_t(\cdot, \hat{f})$ on $\mathcal{R}_t$ for every $(t, \hat{f}) \in T \times \mathcal{S}_{\hat{\Sigma}}$. By an application of Theorem 2.2.1 it now follows that there exists $\hat{f}_* := (\delta_{*t})_{t\in T} \in \mathcal{S}_{\hat{\Sigma}}$ such that $\hat{f}_*(t) \in \mathrm{argmax}_{\delta\in\hat{S}_t} U_t(\delta, \hat{f}_*)$ for every $t \in T$, which is precisely to say that $(\delta_{*t})_{t\in T}$ has the equilibrium property stated in the theorem.

Step 2: general case. The trick is to reduce this case to step 1 by imitating an argument stated on p. 78 of [25]. Let $\mathcal{C}$ be the collection of all sequences $(\mathcal{G}_\tau^0)_{\tau\in T}$, where each $\mathcal{G}_\tau^0$ is a countably generated sub-σ-algebra of $\mathcal{F}_\tau$. The principal point to note is the following identity:

$$\mathcal{E} := \cup\{\Pi_{\tau\in T}\mathcal{G}_\tau^0 \times \mathcal{B}(S) : (\mathcal{G}_\tau^0)_{\tau\in T} \in \mathcal{C}\} = \mathcal{F} \times \mathcal{B}(S). \tag{4.2}$$

Recall here that $\Pi_{\tau\in T}\mathcal{G}_\tau^0 \times \mathcal{B}(S)$ indicates a product σ-algebra. To prove (4.2), observe that $\mathcal{E}$, as defined, is a σ-algebra. [For instance, if (A_m) is a sequence in $\mathcal{E}$, then each A_m belongs to $\Pi_{\tau\in T}\mathcal{G}_\tau^m \times \mathcal{B}(S)$ for some $(\mathcal{G}_\tau^m)_{\tau\in T} \in \mathcal{C}$. But then $\cup_m A_m$ belongs to $\Pi_{\tau\in T}\mathcal{G}_\tau^0 \times \mathcal{B}(S)$, where $\mathcal{G}_\tau^0$ is the σ-algebra generated by $(\mathcal{G}_\tau^m)_{m=1}^\infty$, etc.] This fact immediately proves (4.2). Indeed, one inclusion in (4.2) is trivial, and the other one follows by the fact that for each product set $F := \Pi_{\tau\in T}F_\tau$, with $F_\tau \in \mathcal{F}_\tau$ for all τ, one has $F \in \Pi_{\tau\in T}\mathcal{G}_\tau^0$ with $\mathcal{G}_\tau^0 := \{\emptyset, \Omega_\tau, F_\tau, \Omega_\tau\backslash F_\tau\}$ (observe that such sets F form the generators of $\mathcal{F}$).

Given the fact that the collection $(v_t)_{t\in T}$ is at most countable, (4.2) implies that there exists a sequence $(\mathcal{F}_\tau^0)$ in $\mathcal{C}$ such that for every $t \in T$ the function u_t is $\mathcal{F}^0 \times \mathcal{B}(S)$-measurable, with $\mathcal{F}^0$ defined as the product σ-algebra $\Pi_{\tau\in T}\mathcal{F}_\tau^0$. To see this, observe that it is enough to prove this fact only for one of the v_t, and, actually, to prove it only for a u_t that is of the characteristic function form $u_t = 1_G$, with G a $\mathcal{F} \times \mathcal{B}(S)$-measurable set (indeed, once this is proven, an obvious approximation of the original u_t by step functions gives the entire proof). Since (4.2) implies $G \in \mathcal{E}$, the desired fact for $u_t = 1_G$ already follows. QED

[1] As usual, $(\delta_t, \delta_*^{-t})$ stands for $(\eta_\tau)_{\tau\in T}$ defined by $\eta_t := \delta_t$ and $\eta_\tau := \delta_{*\tau}$ for $\tau \neq t$.

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